

# Universal Irreversibility of Normal Quantum Diffusion

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Time-reversibility measured by the deviation of the perturbed time-reversed motion from the unperturbed one is examined for normal quantum diffusion exhibited by four classes of quantum maps with contrastive physical nature. Irrespective of the systems, there exist a universal minimal quantum threshold above which the system completely loses the past memory, and the time-reversed dynamics as well as the time-reversal characteristics asymptotically trace universal curves independent of the details of the systems.

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*Introduction:* Historically, the irreversible time evolution was considered as characteristics peculiar to macroscopic system composed of many degrees of freedom. However, since the rediscovery of chaos in systems with a small number of degrees of freedom (SND), chaos, which exists in non-integrable dynamical systems, has been considered as a more generic origin of time-irreversibility and dissipation [1].

On the other hand, quantum systems has no counterpart of classical orbits, and chaos in the sense of classical mechanics does not exist. However, in quantum systems which is chaotic in the classical limit, apparently irreversible phenomena such as normal diffusion [2, 3], stationary energy absorption [4] and so on do occur although the number of degrees of freedom is small. Moreover, even in quantum systems with no classical limit, normal diffusion can be realized, as is typically exemplified by more than one-dimensional disordered systems [5]. However, very few works has been done for the direct characterization of time-irreversibility in apparently irreversible quantum systems.

The purpose of this letter is to explore the time-irreversibility of quantum dynamics exhibiting normal diffusion, which is characterized by the stationary linear increase of mean square displacement (MSD),  $M(t) = \sum_x P(x, t)x^2 = Dt$ . (The diffusion space depends on the representation.) It is quite natural to quantify the time irreversibility of normally diffusing systems on the basis of MSD.

We quantitatively characterize the instability underlying quantum dynamics in terms of the sensitivity of time-reversed dynamics to the perturbation applied at a reversal time, which is referred as *time-reversal test*. *Method:* The time-reversal test consists of the following three processes. First the initial point-wise localized state evolves in forward by operating the evolution oper-

ator  $U^t$  until  $t = T$ . At the reversal time  $t = T$ , a perturbation  $\hat{P}(\eta)$  is applied, and finally the perturbed state is evolved in backward by operating the time-reversed evolution operator  $U^{-T}$ . With the use of MSD, the relative irreversibility

$$\mathcal{R}(\eta) = \frac{|M_\eta(2T) - M_0(2T)|}{M_0(T)}, \quad (1)$$

is defined as a function of the perturbation strength  $\eta$ . It is used as a measure of the sensitivity of the quantum dynamics to the external perturbation, which is referred as the *time-reversal characteristics* [9]. We would like to use  $\mathcal{R}$  rather than the "fidelity", which has been frequently used in literatures [6, 7] since [8], as the measure of quantum irreversibility, because  $\mathcal{R}$  allows an immediate comparison between quantum irreversibility and its classical counterpart.

As the perturbation  $\hat{P}$  applied at  $T$ , we mainly use the  $\eta$ -shift operator, which shifts the wavepacket by  $\eta$  in the  $y$ -space canonically conjugate to the diffusion space  $x$ ,

$$\hat{P}_y(\eta) = \exp\{i\eta\hat{x}/\hbar\} = \exp\{\eta\partial/\partial y\}. \quad (2)$$

We call it the "perpendicular shift". There is also the "parallel shift"  $\hat{P}_x(\eta) = \exp\{i\eta\hat{y}/\hbar\}$  which shifts the wavefunction in the diffusion space  $x$  by  $\eta$ , but we use in this letter only "perpendicular shift" on a reason mentioned later. This method is a powerful tool when we measure the instability of quantum dynamics which have no counterpart of the classical orbital instability [9].

*Models:* As the model system for which the time-reversal test is examined, we use four kinds of quantum maps with quite different nature. The models are described by the following common form of unitary operator,

$$\hat{U} = e^{-i\frac{H_0(\hat{p})}{2\hbar}} e^{-i\frac{V(\hat{q})}{\hbar}} e^{-i\frac{H_0(\hat{p})}{2\hbar}}, \quad (3)$$

where  $H_0(\hat{p})$  and  $V(\hat{q})$  represent kinetic energy and potential energy, respectively. Here  $\hat{p}$  and  $\hat{q}$  are momentum and position operators, respectively. The first example is standard map (SM), which is given by  $H_0 =$

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$\frac{p^2}{2}, V(q) = K \cos q$ , is a typical deterministic quantum map whose classical counterpart is chaotic if  $K$  is large enough and shows a nice quantum diffusion in the  $p$  space if  $K \ll 1$  and  $\hbar \ll 1$ . The second example is the perturbed Anderson map with  $H_0(p) = \cos(\frac{p}{\hbar})$  and  $V(q, t) = v_q \{1 + \sum_{i=1}^M \epsilon_i \cos \omega_i t\}$  [10]. It is a quantum map version of Anderson model defined on the discretized lattice  $q \in \mathbf{Z}$ . On-site potential  $v_q$  taking random value uniformly distributed over the range  $[-W, W]$  leads to the localization of wavepacket, but the quasi-periodic perturbation of the strength  $\epsilon$  with the  $M$  incommensurate frequencies destroys the Anderson localization, resulting into a well-behaved diffusion in the  $q$  space if  $M \geq 2$  and  $\epsilon \geq O(1)$ . We call this model the perturbed Anderson map (PAM). In the following numerical calculation, we take  $W = 1.0$  or  $0.5$  and  $\epsilon_i = \frac{\epsilon}{\sqrt{M}}$ , for simplicity, and take incommensurate numbers of  $\omega_i \sim O(1)$  as the frequency set.

The first and the second examples are both intrinsic dynamical systems which contains no stochastic force. We furthermore examine stochastically-perturbed quantum maps as another prototypes showing noise-induced normal diffusion. The third example is the stochastic standard map (SSM)  $\hat{U} = e^{-ip^2/2\hbar} e^{-i\epsilon n_t \sin q/\hbar} e^{-ip^2/2\hbar}$  parametrically driven by the temporal noise  $n_t$  [12]. The final example is a quantum map version of Haken-Strobal model  $\hat{U} = e^{-i \cos(\hat{p}/\hbar)/2\hbar} e^{-in_{qt}/\hbar} e^{-i \cos(\hat{p}/\hbar)/2\hbar}$  driven by spatio-temporal noise [13], which we call Haken map (HM). In both models the applied noise satisfy the sample average  $\langle n_t n_{t'} \rangle = \epsilon_n^2 \delta_{tt'}$  for SSM and  $\langle n_{q,t} n_{q',t'} \rangle = \epsilon_n^2 \delta_{qq'} \delta_{tt'}$  for HM, where  $\epsilon_n$  is strength of the noise.

In the limit  $\hbar \rightarrow 0$ , both SM and SSM have classical counterparts whose orbit is exponentially unstable [14]. However, PAM and HM have not their classical counterparts, since the transfer operator  $i \cos(\hat{p}/\hbar)/2$  has no classical limit. We may expect that quantum normal diffusion systems having a classical limit may somehow mimic the time-irreversibility of the classical counterpart in the limit  $\hbar \rightarrow 0$ . But we have no reference based on the classical theory for PAM and HM, and so we are particularly interested in the time-irreversibility for the latter class of quantum normal diffusion.

*Result:* Figure 1(a) and (b) are typical results of time-reversal test examined for normally diffusing state of SM and PAM. For relatively strong perturbation strength  $\eta$  at the reversed time  $T$ , the time-reversed dynamics follow the unperturbed time-reversed dynamics for only a short period, and it sooner recovers the normal diffusion at the same diffusion constant  $D$  of the forward diffusion (this is due to the time-reversal symmetry of our system). This fact allows us to interpret that the recovered diffusion is delayed from the forward diffusion  $M(t) = Dt$  with the delay time  $\tau_d$ , namely asymptotically  $M_\eta(t) = D(t - \tau_d)$  for  $t \gg T + \tau_d$ . However, as  $\eta$  decreases, there appears a strong tendency of following the unperturbed time-reversed dynamics. Such a nature is

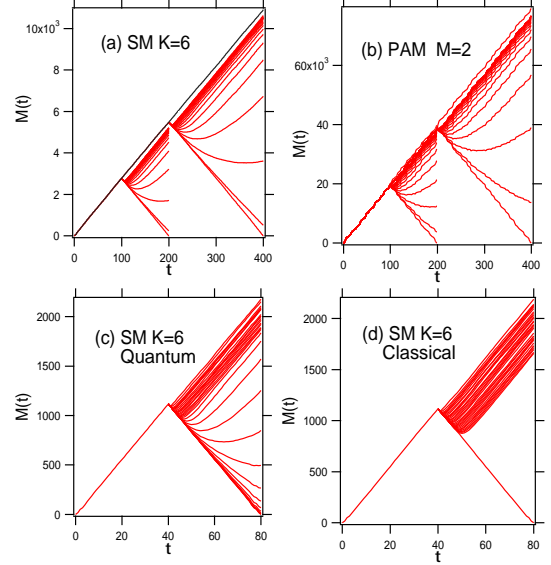


FIG. 1: (Color online) Time-reversal tests of SM and PAM with self-induced normal diffusion. (a) SM with  $K = 6$ ,  $\hbar = \frac{2\pi 121}{2^{17}}$ . (b) PAM with  $M = 2$ ,  $\epsilon = 1.0$ ,  $\hbar = 0.25$ . (c) Short reversed-time version of (a). (d) Classical SM with  $K = 6$  corresponding to (c).

quite different from the classical time-irreversibility typically observed for the classical counterparts of SM, shown in Fig.1(d)

Fig.1(d) and (c) compares the time-reversed dynamics of classical and quantum SM in chaotic regime. In Fig.1(d) the perturbation strength  $\eta$  decreases geometrically, then the delay  $\tau_d$  increase in proportion to  $\eta$ , which means that  $\tau_d \propto \log \eta$ . The  $\log \eta$  dependence of the delay time is the result of the exponential instability inherent in classical chaotic dynamics: the perturbed time-reversed orbit separates from the unperturbed one exponentially, which makes the deviation grow as  $\Delta M_\eta(T, \tau) \equiv M_\eta(T + \tau) - M_0(T + \tau) \sim \eta e^{\lambda \tau}$ , where  $\lambda$  is the Lyapunov exponent. The exponential deviation changes into the delayed diffusion after the nonlinear saturation of exponential instability  $\eta e^{\lambda \tau_d} \sim O(1) (= C)$ , which gives  $\tau_d(\eta) = \frac{\log(C/\eta)}{\lambda}$ . The classical relative irreversibility (1) is thus

$$\mathcal{R}_{cl} \sim 2 - \frac{\tau_d(\eta)}{T} = 2 - \frac{\log C/\eta}{\lambda T}. \quad (4)$$

which agrees with the classical numerical result of Fig.2(a). It means that we have to make  $\eta$  exponentially as small as  $\eta \sim C e^{-\lambda T}$  in order to control the system to restore the time-reversibility. Numerically observed quantum  $\mathcal{R}$  for various  $T$ , is presented in Fig.2. Here  $\eta$  is scaled by a fundamental unit  $\eta_{th}$ , which will be discussed later in detail. It clearly follows the classical  $\log \eta/T$  dependence of Eq.(4) in a relatively large regime of  $\eta$ , but a striking difference between quantum and classical appears in the low  $\eta$ -regime: there exists a

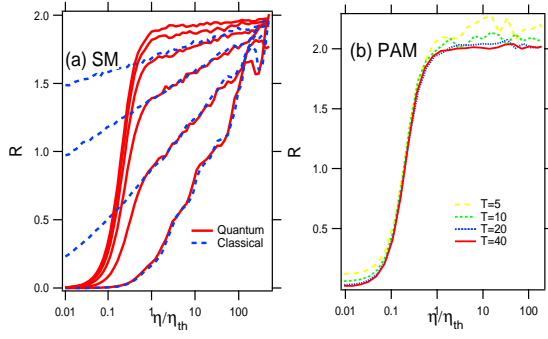


FIG. 2: (Color online) Time-reversal characteristics of (a) quantum (solid) and classical (broken) SM and (b) PAM as functions of the scaled perturbation strength  $\eta/\eta_{th}$  at several reversal times  $T = 5, 10, 20, 40, 80$  from below.  $K = 6$ ,  $\hbar = \frac{2\pi^{121}}{2^{17}}$  for SM and  $W = 1.0, M = 3, \epsilon = 0.5$  for PAM. The classical time-reversal characteristics are well described by Eq.(4).

threshold  $\eta_{th}$  below which the quantum reversed motion, differing entirely from the classical one, restores the time-irreversibility and the relative irreversibility approaches promptly to 0.

The presence of the threshold is a direct manifestation of quantum uncertainty in the quantum time-irreversibility. Suppose that the wavepacket diffuses to cover the range of  $x$  width  $\Delta x(T) = \sqrt{M(T)}$  at the reversal time. So the perpendicular perturbation (2) shifting the quantum state in the  $y(=q)$  space by  $\eta$  sweeps the phase space over the area  $A = \eta\Delta x(T)$ . The shifted quantum state is recognized as classically distinguishable one from the original state if a  $\eta$  is large enough such that the swept area contains more than one quantum state, namely  $A/\hbar > 1$ , which defines the *least quantum perturbation unit* (LQPU),

$$\eta_{th} = \frac{2\pi\hbar}{\Delta X} = \frac{2\pi\hbar}{\sqrt{M(T)}} \quad (5)$$

as the threshold perturbation strength. If  $\eta > \eta_{th}$ , the orbit from the shifted state separates from the orbit from the original state in the classical mechanical way. In Fig.2  $\mathcal{R}$  is displayed as functions of  $\eta$  scaled by the LQPU, and so the quantum threshold is the common value 1 in the unit. Hereafter, we refer the regions  $\eta/\eta_{th} < 1$  and  $\eta/\eta_{th} > 1$  as *quantum region* and *post quantum region*, respectively [15]. The SSM, which has the classical counterpart, shows quite similar behavior as in Fig.2(a).

A typical example of time-reversal characteristics in systems without the classical counterpart is shown in Fig.2(b). Even in this case we can recognize the quantum and the post-quantum regions similarly to Fig.2(a), but we can not observe a significant  $T$  dependence of the characteristics, and convergence to the asymptotic characteristics occurs much more rapidly than SM and SMS. In spite of the difference in the way of convergence, the asymptotic limit exhibits a remarkable universality

irrespective of the kind of the system independent on whether the system has the classical unstable limit or not, and whether the diffusion is due to the stochastic perturbation or not.

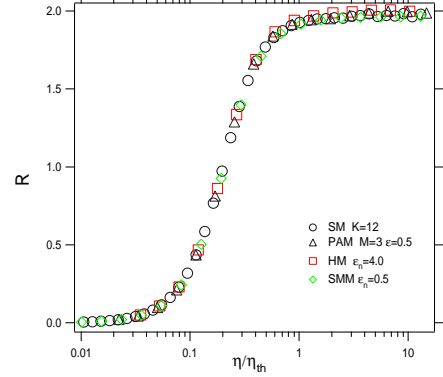


FIG. 3: (Color online) Time-reversal characteristics as a function of the scaled perturbation strength for some quantum maps which exhibit normal diffusion, i.e. SM with  $K = 4.2, 12$  at  $T = 800$ , PAM with  $M = 3, \epsilon = 0.5, 0.8$  at  $T = 200$ , HM with  $\epsilon_n = 4.0$  at  $T = 500$ , and SSM with  $\epsilon_n = 0.1$  at  $T = 3200$ .

Figure 3 depicts the relative irreversibility  $\mathcal{R}$  as a function of the scaled perturbation strength for the four kinds of models, which is obtained in the large limit of  $T$ . All the plots are on a common universal curve, and so the irreversibility  $\mathcal{R}$  reaches to 2 in the same way as  $\eta/\eta_{th}$  exceeds 1, which means that the system completely lose the memory and reset to the stationary diffusion beyond  $\eta_{th}$  in a universal way. It is the significance of the universal quantum threshold  $\eta_{th}$  as the LQPU.

It seems that such a class of universality is not only a feature of the irreversibility defined at the returning time  $t = 2T$  but also is a general property of the time-reversed dynamics itself. Let us return to the separation of the perturbed time-reversed process from the unperturbed one, which is represented by the difference  $\Delta M_\eta(T, \tau) = M_\eta(\tau+T) - M_0(\tau+T)$ . In Fig.4, the scaled separation  $\Delta M_\eta(T, \tau)/M_0(T)$  is shown as a function of the scaled time  $\tau/T$  for SM and PAM. The observed scaled separation  $\Delta M_\eta(T, \tau)/M_0(T)$  vs scaled time  $\tau/T$  is on the common curve independent of the system in the limit  $T \rightarrow \infty$  if the scaled perturbation strength  $\eta/\eta_{th}$  is the same. Such a universal behavior can be explained if we admit the universality of the time-reversal characteristics  $\mathcal{R}$  demonstrated by Fig.3, namely

$$\mathcal{R} = F\left(\frac{\eta}{\eta_{th}(T)}\right). \quad (6)$$

Next we suppose the stationarity of the time-reversed dynamics, which means that for the same  $\eta$  the difference  $\Delta M_\eta \equiv M_\eta(T+\tau) - M_0(T+\tau)$  does not depend on the reversal time  $T$ , i.e.,

$$\Delta M_\eta(T, \tau) = G(\eta, \tau), \quad (7)$$

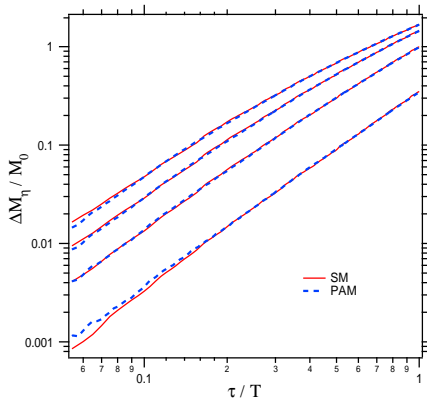


FIG. 4: (Color online) Log-log plots of the scaled separation  $\Delta M_\eta(T, \tau)/M_0(T)$  as a function of scaled time  $\tau/T$  for several scaled perturbation strength  $\eta/\eta_{th} = 0.1, 0.2, 0.3, 0.4$  from below at reversal time  $T = 400$ . The data are for SM with  $K = 12$  and  $\hbar = \frac{2\pi 975}{2^{20}}$  and PAM with  $M = 3, \epsilon = 0.5$ .

where  $G$  is a function depending only on  $\eta$  and  $\tau$ . We do not give explicit evidences for the stationarity hypothesis, however, extended numerical examination supports the validity of this plausible hypothesis. Equations (6) and (7) claim that  $G(\eta, \tau) = D\tau F(\frac{\eta}{\eta_{th}(\tau)})$ , which is immediately followed by the relation

$$\mathcal{R} = \frac{\tau}{T} F\left(\frac{\eta}{\eta_{th}(T)} \left\{\frac{\tau}{T}\right\}^\chi\right), \quad (8)$$

where  $\eta_{th}(T) \propto T^{-\chi}$ . The index  $\chi$  is determined by the type of perturbation as,  $\chi = 1/2$  for perpendicular  $\eta$ -shift and  $\chi = 0$  for parallel  $\eta$ -shift. Thus,  $\Delta M_\eta(T, \tau)/M_0(T)$  is determined only by the scaled perturbation strength  $\eta/\eta_{th}$  and the scaled time  $\tau/T$  for the reversal time  $T(> T_{th})$ .

*In conclusion:* The scaled universal relation (6) implies  $\mathcal{R} = 2$  in the post quantum region, which means the complete loss of initial memory in that region, but in the

quantum region it reaches zero in a common way independent of the reversal time  $T$ . The latter feature reveals an excellent stability and memory effect of the quantum systems against the perturbation in the quantum region, which can never be expected for their classical counterparts. On the contrary, the stationarity of the system means that the time-reversed dynamics is free from the past memory at the very time at which the time-reversed operation is applied. These two apparently contradictory features are combined to yield the universal scaled time-reversed dynamics in quantum normal diffusion.

Quantum characteristics of time-reversibility has been explored using four kinds of quantum maps exhibiting well-behaved normal diffusion. In spite of the quite different nature of four models, loss of initial memory is controlled by an universal quantum parameter called the least quantum perturbation unit(LQPU), and the time-reversed dynamics as well as time-reversal characteristics is universal independent of the details of quantum maps.

We conjecture that the universal time irreversible characteristics demonstrated here is commonly shared by the “most irreversible” class of quantum systems which require the most delicate control in order to retrieve the initial memory. It should be stressed that such a universal feature is observed only for normal diffusion systems, and the time-reversibility of localizing system and delocalized system without normal diffusion deviates much from the universal behavior in a non-universal way [12]. The quantum systems exhibiting the normal diffusion would be the most unstable class of simple quantum systems which models “quantum irreversibility”. Moreover, to clarify the quantum irreversibility would provide basic knowledge for the ultimate origin of intrinsic dissipation of small quantum systems.

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